arXiv:nlin/0010057v2 [nlin.SI] 5 Feb 2001

Bethe vectors of the $osp(1|2)$ Gaudin model

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Abstract

The eigenvectors of the $osp(1|2)$ invariant Gaudin hamiltonians are found using explicitly constructed creation operators. Commutation relations between the creation operators and the generators of the loop superalgebra are calculated. The coordinate representation of the Bethe states is presented. The relation between the Bethe vectors and solutions to the Knizhnik-Zamolodchikov equation yields the norm of the eigenvectors.

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1 Introduction

Classifying integrable systems solvable in the framework of the quantum inverse scattering method [1, 2] by underlying dynamical symmetry algebras, one could say that the Gaudin models are the simplest ones being related to loop algebras and classical r-matrices. More sophisticated solvable models correspond to Yangians, quantum affine algebras, elliptic quantum groups, etc.

Gaudin models [3] are related to classical r-matrices, and the density of Gaudin hamiltonians

$$H^{(a)} = \sum_{b \neq a} r_{ab}(z_a - z_b) \tag{1.1}$$

coincides with the r-matrix. Condition of their commutativity $[H^{(a)}, H^{(b)}] = 0$ is nothing else but the classical Yang-Baxter equation (YBE).

The Gaudin models (GM) associated to classical r-matrices of simple Lie algebras were studied in many papers (see [3-9] and references therein). The spectrum and eigenfunctions were found using different methods (coordinate and algebraic Bethe Ansatz [3, 4], separated variables [5], etc.). A relation to the Knizhnik-Zamolodchikov (KZ) equation of conformal filed theory was established [7-9].

There are additional peculiarities of Gaudin models related to classical r-matrices based on Lie superalgebras due to Z_2-grading of representation spaces and operators. The study of the $osp(1|2)$ invariant Gaudin model corresponding to the simplest nontrivial super-case of the $osp(1|2)$ invariant r-matrix [10] started in [11]. The spectrum of the $osp(1|2)$ invariant Gaudin hamiltonians $H^{(a)}$ was given, antisymmetry property of their eigenstates was claimed, and a two site model was connected with some physically interesting one (a Dicke model).

The creation operators used in the $sl(2)$ GM coincide with one of the L-matrix entry [3, 4]. However, in the $osp(1|2)$ case, as we will show, the creation operators are complicated polynomials of two generators $X^+(\lambda)$ and $v^+(\mu)$ of the loop superalgebra. We introduce B-operators by a recurrence relation (Section 3). Acting on the lowest spin vector the B-operators generate exact eigenstates of the Gaudin hamiltonians $H^{(a)}$, provided Bethe equations are imposed on parameters of the states. Furthermore, the recurrence relation is solved explicitly and the commutation relations between the B-operators and the generators of the loop superalgebra $\mathcal{L}(osp(1|2))$ are calculated. We prove that the constructed states are lowest spin vectors of the global finite dimensional superalgebra $osp(1|2)$, as it is the case for many invariant quantum integrable models [12]. Moreover, a striking coincidence between the spectrum of the $osp(1|2)$ invariant Gaudin hamiltonians of spin s and the spectrum of the hamiltonians of the $sl(2)$ GM of the integer spin $2s$ is found (Section 3).

A connection between the B-states, when the Bethe equations are not imposed on their parameters, of the Gaudin models for simple Lie algebras to the solutions to the Knizhnik-Zamolodchikov equation was established in the papers [7, 8]. An explanation of this connection based on Wakimoto modules at critical level of the underlying affine algebra was given in [8]. An explicit form of the Bethe vectors in the coordinate representation was given in both papers [7, 8]. The coordinate Bethe Ansatz for the B-states of the $osp(1|2)$ GM is obtained in our paper as well. Using commutation relations between the B-operators and the transfer matrix $t(\lambda)$ we demonstrate algebraically that explicitly constructed B-states yield a solution to the KZ equation (Section 4). This con-

nection permits us to calculate the norm of the eigenstates of the Gaudin hamiltonians. An analogous connection is expected between quantum $osp(1|2)$ spin system related to the graded YBE [10, 13] and quantum KZ equation following the lines of [14].

The norm and correlation functions of the $sl(2)$ invariant GM were evaluated in [5] using Gauss factorization of a group element and Riemann-Hilbert problem. The study of this problem for the GM based on the $osp(1|2)$ Lie superalgebra is in progress. Nevertheless we propose a formula for the scalar products of the Bethe states which is analogous to the $sl(2)$ case (Conclusion).

2 $OSp(1|2)$-invariant R-matrix

Many properties of the Gaudin models can be obtained as a quasi-classical limit of the corresponding quantum spin systems related to solutions $R(\lambda;\eta)$ to the YBE. In the quasi-classical limit $\eta \to 0$

$$R(\lambda;\eta) = I + \eta r(\lambda) + \mathcal{O}(\eta^2),$$

some relations simplify and therefore can be solved explicitly providing more detailed results for the GM.

The graded Yang-Baxter equation [2, 10] differs from the usual YBE by some sign factors due to the embedding of R-matrix into the space of matrices acting on the Z_2-graded tensor product $V_1 \otimes V_2 \otimes V_3$. At this point our aim is to reach fundamental $osp(1|2)$ invariant solution. The rank of the orthosymplectic Lie algebra $osp(1|2)$ is one and its dimension is five. The three even generators are h, X^+, X^- and the two odd generators are v^+, v^- [17]. The (graded) commutation relations between the generators are

$$\begin{array}{cc} [h, X^{\pm}] = \pm 2X^{\pm}\,, & [X^+, X^-] = h\,, \\ [h, v^{\pm}] = \pm v^{\pm}\,, & [v^+, v^-]_+ = -h\,, \\ [X^{\mp}, v^{\pm}] = v^{\mp}\,, & [v^{\pm}, v^{\pm}]_+ = \pm 2X^{\pm}\,, \end{array} \tag{2.1}$$

together with $[X^{\pm}, v^{\pm}] = 0$. Notice that the generators h and $v^{\pm}$ considered here (2.1) differ by a factor of 2 from the ones used in [17, 10]. Thus the Casimir element is

$$\begin{aligned} c_2 &= h^2 + 2\left(X^+X^- + X^-X^+\right) + \left(v^+v^- - v^-v^+\right) \\ &= h^2 - h + 4X^+X^- + 2v^+v^-\,. \end{aligned} \tag{2.2}$$

For further comparison with the $sl(2)$ Gaudin model [3, 4] and due to the chosen set of generators (2.1) we parameterize the finite dimensional irreducible representations $V^{(l)}$ of the $osp(1|2)$ Lie superalgebra by an integer l, so that their dimensions $2l+1$ and the values of the Casimir element (2.2) $c_2 = l(l+1)$ coincide with the same characteristics of the integer spin l irreducible representations of $sl(2)$.

The fundamental irreducible representation V of $osp(1|2)$ is three dimensional. We choose a gradation of the basis vectors $e_j;\, j = 1,2,3$ to be $(0,1,0)$.

The invariant R-matrix is a linear combination [10]

$$R = \lambda\left(\lambda + \frac{3\eta}{2}\right) I + \eta\left(\lambda + \frac{3\eta}{2}\right)\mathcal{P} - \eta\,\lambda\,K\,, \tag{2.3}$$

of the three $OSp(1|2)$ group invariant operators $[g \otimes g, X] = 0$, $g \in OSp(1|2)$, $X \in \mathrm{End}\,(V \otimes V)$, acting on $V \otimes V$: the identity I, the graded permutation $\mathcal{P}$ and a

rank one projector K. In the equation (2.3) λ is the spectral parameter, and η is a quasi-classical parameter.

The L-operator of the quantum spin system on a one-dimensional lattice with N sites coincides with R-matrix acting on a tensor product $V_0 \otimes V_a$ of auxiliary space V_0 and the space of states at site $a = 1, 2, \ldots N$

$$L_{0a}(\lambda - z_a) = R_{0a}(\lambda - z_a)\,, \tag{2.4}$$

where z_a is a parameter of inhomogeneity (site dependence) [1, 2]. Corresponding monodromy matrix T is an ordered product of the L-operators

$$T(\lambda; \{z_a\}_1^N) = L_{0N}(\lambda - z_N) \ldots L_{01}(\lambda - z_1) = \prod_{\substack{a=1 \\ \longleftarrow}}^{N} L_{0a}(\lambda - z_a)\,. \tag{2.5}$$

The commutation relations of the T-matrix entries follow from the FRT-relation [1]

$$R_{12}(\lambda - \mu) T_1(\lambda) T_2(\mu) = T_2(\mu) T_1(\lambda) R_{12}(\lambda - \mu)\,. \tag{2.6}$$

Multiplying (2.6) by R_{12}^{-1} and taking the super-trace over $V_1 \otimes V_2$, one gets commutativity of the transfer matrix

$$t(\lambda) = \sum_j (-1)^{j+1} T_{jj}(\lambda; \{z_a\}_1^N) = T_{11} - T_{22} + T_{33} \tag{2.7}$$

for different values of the spectral parameter $t(\lambda)t(\mu) = t(\mu)t(\lambda)$.

The choice of the L-operators (2.4) corresponds to the following space of states of the $osp(1|2)$-spin system

$$\mathcal{H} = \overset{N}{\underset{a=1}{\otimes}} V_a\,.$$

The eigenvalue of the transfer matrix $t(\lambda)$ in this space is [10]

$$\begin{aligned} \Lambda(\lambda; \{\mu_j\}_1^M) &= \alpha_1^{(N)}(\lambda; \{z_a\}_1^N) \prod_{j=1}^{M} S_1(\lambda - \mu_j) - \alpha_2^{(N)}(\lambda; \{z_a\}_1^N) \times \\ &\times \prod_{j=1}^{M} S_1\left(\lambda - \mu_j + \frac{\eta}{2}\right) S_{-1}(\lambda - \mu_j + \eta) + \\ &+ \alpha_3^{(N)}(\lambda; \{z_a\}_1^N) \prod_{j=1}^{M} S_{-1}\left(\lambda - \mu_j + \frac{3\eta}{2}\right)\,, \end{aligned} \tag{2.8}$$

where $\alpha_j^{(N)}(\lambda; \{z_a\}_1^N) = \prod_{b=1}^N \alpha_j(\lambda - z_b)\,; j = 1, 2, 3\,,$

$$\begin{aligned} \alpha_1(\lambda) &= (\lambda + \eta)\,(\lambda + 3\eta/2)\,, \quad \alpha_2(\lambda) = \lambda\,(\lambda + 3\eta/2)\,, \\ \alpha_3(\lambda) &= \lambda\,(\lambda + \eta/2)\,, \quad S_n(\mu) = \frac{\mu - n\eta/2}{\mu + n\eta/2}\,. \end{aligned} \tag{2.9}$$

Although according to (2.8) the eigenvalue has formally two sets of poles at $\lambda = \mu_j - \eta/2$ and $\lambda = \mu_j - \eta$, the corresponding residues are zero due to the Bethe equations [10]

$$\prod_{a=1}^{N} \left(\frac{\mu_j - z_a + \eta/2}{\mu_j - z_a - \eta/2}\right) = \prod_{k=1}^{M} S_1(\mu_j - \mu_k) S_{-2}(\mu_j - \mu_k)\,. \tag{2.10}$$

If we take different spins l_a at different sites of the lattice and the following space of states

$$\mathcal{H} = \overset{N}{\underset{a=1}{\otimes}} V_a^{(l_a)} ,$$

then the factors on the left hand side of (2.10) will be spin dependent too

$$\frac{\mu_j - z_a + \eta l_a/2}{\mu_j - z_a - \eta l_a/2} .$$

The $osp(1|2)$ invariant R-matrix (2.3) has more complicated structure than the $sl(2)$ invariant R-matrix of C. N. Yang $R = \lambda I + \eta \mathcal{P}$. As a consequence the commutation relations of the entries $T_{ij}(\lambda)$ of the T-matrix (2.5) are more complicated and construction of the eigenstates of the transfer matrix $t(\lambda)$ by the algebraic Bethe Ansatz can be done only using a complicated recurrence relation expressed in terms of $T_{ij}(\mu_k)$ [16] (see also [13] for the case of $osp(1|2)$). It will be shown below that due to a simplification of this relation in the quasi-classical limit $\eta \to 0$ one can solve it and find the creation operators for the $osp(1|2)$ Gaudin model explicitly.

3 $OSp(1|2)$ Gaudin model and creation operators

The classical r-matrix of the orthosymplectic Lie superalgebra $osp(1|2)$ can be expressed in a pure algebraic form using Casimir element in the tensor product $osp(1|2) \otimes osp(1|2)$ [10]

$$\hat{r}\,(\lambda) = \frac{1}{\lambda} c_2^{\otimes} , \tag{3.1}$$

where $c_2^{\otimes} = h \otimes h + 2\,(X^+ \otimes X^- + X^- \otimes X^+) + (v^+ \otimes v^- - v^- \otimes v^+)$. The matrix form of the Casimir element $\hat{r}$ in the fundamental representation π of $osp(1|2)$ follows from (3.1) by substituting appropriate 3×3 matrices instead of the $osp(1|2)$ generators (2.1) and taking into account Z_2-graded tensor product of even and odd matrices [10]. Alternatively, the same matrix form of $\hat{r}$ can be obtained as a term linear in η in the quasi-classical expansion of (2.3)

$$r(\lambda) = \frac{r_0}{\lambda} = \frac{1}{\lambda}\,(\mathcal{P} - K) ,$$

where $\mathcal{P}$ is a graded permutation matrix and K is a rank one projector.

A quasi-classical limit $\eta \to 0$ of the FRT-relations (2.6) results in a matrix form of the loop superalgebra relation $(T(\lambda;\eta) = I + \eta L(\lambda) + \mathcal{O}(\eta^2))$

$$\left[\underset{1}{L}\,(\lambda),\, \underset{2}{L}\,(\mu)\right] = -\left[r_{12}(\lambda-\mu)\,,\, \underset{1}{L}\,(\lambda) + \underset{2}{L}\,(\mu)\right] . \tag{3.2}$$

Both sides of this relation have the usual commutators of even 9×9 matrices $\underset{1}{L}\,(\lambda) = L(\lambda) \otimes I_3$, $\underset{2}{L}\,(\mu) = I_3 \otimes L(\mu)$ and $r_{12}(\lambda-\mu)$, where I_3 is 3×3 unit matrix and $L(\lambda)$ has loop superalgebra valued entries:

$$L(\lambda) = \begin{pmatrix} h(\lambda) & -v^-(\lambda) & 2X^-(\lambda) \\ v^+(\lambda) & 0 & v^-(\lambda) \\ 2X^+(\lambda) & v^+(\lambda) & -h(\lambda) \end{pmatrix} \tag{3.3}$$

The relation (3.2) is a compact matrix form of the following commutation relations between the generators $h(\lambda)$, $v^{\pm}(\mu)$, $X^{\pm}(\nu)$ of the loop superalgebra under consideration

$$
\begin{aligned}
\big[h(\lambda)\,,\; X^{\pm}(\mu)\big] &= \mp 2\frac{X^{\pm}(\lambda)-X^{\pm}(\mu)}{\lambda-\mu} & \big[h(\lambda)\,,\; v^{\pm}(\mu)\big] &= \mp\frac{v^{\pm}(\lambda)-v^{\pm}(\mu)}{\lambda-\mu}\,,\\
\big[X^{+}(\lambda)\,,\; X^{-}(\mu)\big] &= -\frac{h(\lambda)-h(\mu)}{\lambda-\mu} & \big[X^{\pm}(\lambda)\,,\; v^{\mp}(\mu)\big] &= -\frac{v^{\pm}(\lambda)-v^{\pm}(\mu)}{\lambda-\mu}\,,\\
\big[v^{+}(\lambda)\,,\; v^{-}(\mu)\big]_{+} &= \frac{h(\lambda)-h(\mu)}{\lambda-\mu} & \big[v^{\pm}(\lambda)\,,\; v^{\pm}(\mu)\big]_{+} &= \mp 2\frac{X^{\pm}(\lambda)-X^{\pm}(\mu)}{\lambda-\mu}\,,
\end{aligned}
\tag{3.4}
$$

together with $[X^{\pm}(\lambda)\,,\; v^{\pm}(\mu)] = 0\,.$

These commutation relations (3.4) define the positive part $\mathcal{L}_{+}(osp(1|2))$ of the loop superalgebra. The usual generators Y_n of a loop algebra parameterized by non-negative integer, are obtained from the expansion $Y(\lambda) = \sum_{n\geq 0} Y_n\, \lambda^{-(n+1)}$. In particular, taking all $Y_n = 0$ for $n > 0$ one gets an L-operator $L(\lambda) = L_0\, \lambda^{-1}$, where L_0 is $osp(1|2)$-valued matrix. This L_0 similar to r_0 satisfies cubic characteristic equation with the $osp(1|2)$ Casimir element (2.2) as coefficient

$$
L_0^3 + 2L_0^2 - (c_2-1)\,L_0 - c_2 I = 0\,. \tag{3.5}
$$

A Gaudin realization of the loop algebra (3.4) can be defined through the generators $Y = (h, v^{\pm}, X^{\pm})$

$$
Y(\lambda) = \sum_{a=1}^{N}\frac{Y_a}{\lambda - z_a}\,, \qquad Y_a \in \mathrm{End}\,(V_a)\,, \tag{3.6}
$$

where Y_a are $osp(1|2)$ generators in an irreducible representation $V_a^{(l_a)}$ of the lowest spin $-l_a$ associated with each site a [3, 4]. Then the L-operator (3.3) has the form

$$
L(\lambda;\{z_a\}_1^N) \;=\; \sum_{a=1}^{N}\frac{L_a}{\lambda - z_a}\,, \tag{3.7}
$$

here $\{z_a\}_1^N$ are parameters of the model (cf (2.5)). It follows from the relation (3.7) that the first term in the asymptotic expansion near $\lambda = \infty$ defines generators of the global superalgebra $osp(1|2) \subset \mathcal{L}_{+}(osp(1|2))$

$$
L_{gl} = \lim_{\lambda\to\infty} \lambda\, L(\lambda) \;=\; \sum_{a=1}^{N} L_a\,, \tag{3.8}
$$

with h_{gl}, $v_{gl}^{\pm}$ and $X_{gl}^{\pm}$ as generators and entries of L_{gl} (cf (3.3)). Moreover, from the equation (3.2) we get

$$
\Big[\underset{1}{L_{gl}},\, \underset{2}{L}\,(\mu)\Big] = -\Big[r_0\,,\, \underset{2}{L}\,(\mu)\Big]\,, \tag{3.9}
$$

here $\underset{1}{L_{gl}} = L_{gl}\otimes I_3$, *e.g.* $[h_{gl}\,,\, v^{+}(\mu)] = v^{+}(\mu)\,.$

Let us consider the loop superalgebra $\mathcal{L}_+(osp(1|2))$ as the dynamical symmetry algebra, *i.e.* as the algebra of observables. In order to define a dynamical system besides the algebra of observables we need to specify a hamiltonian. It is a well-known fact that due to the r-matrix relation (3.2), the so-called Sklyanin linear brackets, the elements

$$\begin{aligned} t(\lambda) &= \frac{1}{2}\,\mathrm{str}\, L^2(\lambda) = h^2(\lambda) + 2[X^+(\lambda)\,,\, X^-(\lambda)]_+ + [v^+(\lambda)\,,\, v^-(\lambda)]_- \\ &= h^2(\lambda) + h'(\lambda) + 4X^+(\lambda)X^-(\lambda) + 2v^+(\lambda)v^-(\lambda) \end{aligned} \tag{3.10}$$

commute for different values of the spectral parameter $t(\lambda)t(\mu) = t(\mu)t(\lambda)$. Thus, $t(\lambda)$ can be considered as a generating function of integrals of motion.

It is straightforward to calculate the commutation relations between the operator $t(\lambda)$ and the generators of the loop algebra $X^+(\mu)$ and $v^+(\mu)$

$$\big[t(\lambda), X^+(\mu)\big] = 4\frac{X^+(\mu)h(\lambda) - X^+(\lambda)h(\mu)}{\lambda-\mu} - \frac{v^+(\lambda)v^+(\mu) - v^+(\mu)v^+(\lambda)}{\lambda-\mu}\,, \tag{3.11}$$

$$\big[t(\lambda), v^+(\mu)\big] = 2\frac{v^+(\mu)h(\lambda) - v^+(\lambda)h(\mu)}{\lambda-\mu} - 4\frac{X^+(\mu)v^-(\lambda) - X^+(\lambda)v^-(\mu)}{\lambda-\mu}\,. \tag{3.12}$$

A direct consequence of the equation (3.9) is an invariance of the generating function of integrals of motion $t(\lambda)$ under the action of the global $osp(1|2)$ $[t(\lambda), L_{gl}] = 0$.

We can consider the representation space $\mathcal{H}_{ph}$ of the dynamical algebra to be a lowest spin $\rho(\lambda)$ representation of the loop superalgebra with the lowest spin vector Ω_-

$$h(\lambda)\Omega_- = \rho(\lambda)\Omega_-\,, \quad v^-(\lambda)\Omega_- = 0\,. \tag{3.13}$$

In particular, a representation of the Gaudin realization (3.7) can be obtained by considering irreducible representations $V_a^{(l_a)}$ of the Lie superalgebra $osp(1|2)$ defined by a spin $-l_a$ and a lowest spin vector ω_a such that $v_a^-\omega_a = 0$ and $h_a\omega_a = -l_a\omega_a$. Thus,

$$\Omega_- = \overset{N}{\underset{a=1}{\otimes}} \omega_a\,, \quad \text{and} \quad \rho(\lambda) = \sum_{a=1}^{N} \frac{-l_a}{\lambda - z_a}\,. \tag{3.14}$$

It is a well-known fact in the theory of GM [3, 4], that the Gaudin hamiltonian

$$H^{(a)} = \sum_{b\neq a} \frac{c_2^{\otimes}(a,b)}{z_a - z_b}\,, \tag{3.15}$$

here $c_2^{\otimes}(a,b) = h_a h_b + 2\left(X_a^+X_b^- + X_a^-X_b^+\right) + \left(v_a^+v_b^- - v_a^-v_b^+\right)$, can be obtained as the residue of the operator $t(\lambda)$ at the point $\lambda = z_a$ using the expansion

$$t(\lambda) = \sum_{a=1}^{N}\left(\frac{l_a(l_a+1)}{(\lambda-z_a)^2} + 2\,\frac{H^{(a)}}{\lambda - z_a}\right). \tag{3.16}$$

To construct the set of eigenstates of the generating function of integrals of motion $t(\lambda)$ we have to define appropriate creation operators. The creation operators used in the $sl(2)$ Gaudin model coincide with one of the L-matrix entry [3, 4]. However, in the $osp(1|2)$ case the creation operators are complicated functions of the two generators $X^+(\lambda)$ and $v^+(\mu)$ of the loop superalgebra.

Definition 3.1 *Let* $B_M(\mu_1,\dots,\mu_M)$ *belong to the Borel subalgebra of the* $osp(1|2)$ *loop superalgebra* $\mathcal{L}_+(osp(1|2))$ *such that*

$$B_M(\mu_1,\dots,\mu_M)=v^+(\mu_1)B_{M-1}(\mu_2,\dots,\mu_M)+2X^+(\mu_1)\sum_{j=2}^{M}\frac{(-1)^j}{\mu_1-\mu_j}B^{(j)}_{M-2}(\mu_2,\dots,\mu_M)\,, \tag{3.17}$$

with $B_0=1$, $B_1(\mu)=v^+(\mu)$ *and* $B_M=0$ *for* $M<0$. *The notation* $B^{(j)}_{M-2}(\mu_2,\dots,\mu_M)$ *means that the argument* μ_j *is omitted.*

As we will show below, the B-operators are such that the Bethe vectors are generated by their action on the lowest spin vector Ω_- (3.13). To prove this result we will need some important properties of the B-operators. All the properties of the creation operators $B_M(\mu_1,\dots,\mu_M)$ listed below can be demonstrated by induction method. Since their proofs are lengthy and do not contain illuminating insights we will omit them.

1 *The creation operators* $B_M(\mu_1,\dots,\mu_M)$ *are antisymmetric functions of their arguments*

$$B_M(\mu_1,\dots,\mu_k,\mu_{k+1},\dots,\mu_M)\;=\;-\,B_M(\mu_1,\dots,\mu_{k+1},\mu_k,\dots,\mu_M)\,, \tag{3.18}$$

here $1\le k<M$ *and* $M\ge 2$.

Subsequently we calculate the commutation relations between the generators of the loop superalgebra $\mathcal{L}_+(osp(1|2))$ and the B-operators. In order to simplify the formulas we will omit the arguments and denote the creation operator $B_M(\mu_1,\dots,\mu_M)$ by B_M.

2 *The commutation relations between the creation operator* B_M *and the generators* $v^+(\lambda)$, $h(\lambda)$, $v^-(\lambda)$ *of the loop superalgebra are given by*

$$v^+(\lambda)B_M=(-1)^M B_M v^+(\lambda)+2\sum_{j=1}^{M}(-1)^j\frac{X^+(\lambda)-X^+(\mu_j)}{\lambda-\mu_j}\,B^{(j)}_{M-1}\,, \tag{3.19}$$

$$\begin{aligned} h(\lambda)B_M \;&=\; B_M\left(h(\lambda)+\sum_{i=1}^{M}\frac{1}{\lambda-\mu_i}\right)+\sum_{i=1}^{M}\frac{(-1)^i}{\lambda-\mu_i}\times\\ &\times\left(v^+(\lambda)B^{(i)}_{M-1}+2X^+(\lambda)\sum_{j\neq i}^{M}\frac{(-1)^{j+\Theta(i-j)}}{\mu_i-\mu_j}B^{(i,j)}_{M-2}\right)\,, \end{aligned} \tag{3.20}$$

$$\begin{aligned} v^-(\lambda)B_M \;&=\; (-1)^M B_M v^-(\lambda)+\sum_{j=1}^{M}\left(\frac{h(\lambda)-h(\mu_j)}{\lambda-\mu_j}+\sum_{k\neq j}^{M}\frac{1}{(\lambda-\mu_k)(\mu_k-\mu_j)}\right)\times\\ &\times\;(-1)^{j-1}B^{(j)}_{M-1}+v^+(\lambda)\sum_{i<j}^{M}(-1)^{i-j-1}\frac{B^{(i,j)}_{M-2}}{\mu_i-\mu_j}\left(\frac{1}{\lambda-\mu_i}+\frac{1}{\lambda-\mu_j}\right)\,. \end{aligned} \tag{3.21}$$

here the upper index of $B^{(j)}_{M-1}$ *means that the argument* μ_j *is omitted, the upper index of* $B^{(i,j)}_{M-2}$ *means that the parameters* μ_i,μ_j *are omitted and* $\Theta(j)$ *is Heaviside function.*

Already at this point we can make some useful observations.

Remark 3.1 *The commutation relations between the generators of the global* $osp(1|2)$ *and the* B*-operators follow from the previous property. To see this we multiply (3.19), (3.20) and (3.21) by* λ *and then take the limit* $\lambda\to\infty$*. In this way we obtain*

$$v^+_{gl}B_M = (-1)^M B_M v^+_{gl} - 2\sum_{j=1}^{M}(-1)^j X^+(\mu_j)B^{(j)}_{M-1}\,, \tag{3.22}$$

$$h_{gl}B_M = B_M\left(h_{gl}+M\right)\,, \tag{3.23}$$

$$v^-_{gl}B_M = (-1)^M B_M\, v^-_{gl} + \sum_{j=1}^{M}(-1)^j B^{(j)}_{M-1}\left(h(\mu_j)+\sum_{k\neq j}^{M}\frac{1}{\mu_j-\mu_k}\right)\,. \tag{3.24}$$

The proof of the main theorem is based on subsequent property of the creation operators.

3 *The generating function of integrals of motion* $t(\lambda)$ *(3.10) has the following commutation relation with the creation operator* $B_M(\mu_1,\ldots,\mu_M)$

$$\begin{aligned} t(\lambda)B_M &= B_M t(\lambda) + 2B_M\left(h(\lambda)\sum_{i=1}^{M}\frac{1}{\lambda-\mu_i}+\sum_{i<j}^{M}\frac{1}{(\lambda-\mu_i)(\lambda-\mu_j)}\right)\\ &+ 2\sum_{i=1}^{M}\frac{(-1)^i}{\lambda-\mu_i}\left(v^+(\lambda)B^{(i)}_{M-1}+2X^+(\lambda)\sum_{j\neq i}^{M}\frac{(-1)^{j+\Theta(i-j)}}{\mu_i-\mu_j}B^{(i,j)}_{M-2}\right)\widehat{\beta}_M(\mu_i)\\ &+ 4\sum_{i=1}^{M}\frac{(-1)^i}{\lambda-\mu_i}B^{(i)}_{M-1}\left(X^+(\lambda)v^-(\mu_i)-X^+(\mu_i)v^-(\lambda)\right)\,. \end{aligned} \tag{3.25}$$

The notation we use here for the operator $\widehat{\beta}_M(\mu_i)=h(\mu_i)+\sum_{j\neq i}^{M}(\mu_i-\mu_j)^{-1}$.

In the Gaudin realization (3.6) the creation operators $B_M(\mu_1,\ldots,\mu_M)$ have some specific analytical properties.

4 *The* B*-operators in the Gaudin realization (3.6) satisfy an important differential identity*

$$\frac{\partial}{\partial z_a}B_M=\sum_{j=1}^{M}\frac{\partial}{\partial\mu_j}\left(\frac{(-1)^j}{\mu_j-z_a}\left(v^+_a B^{(j)}_{M-1}+2\,X^+_a\sum_{k\neq j}^{M}\frac{(-1)^{k+\Theta(j-k)}}{\mu_j-\mu_k}B^{(j,k)}_{M-2}\right)\right)\,. \tag{3.26}$$

This identity will be fundamental step in establishing a connection between the Bethe vectors and the KZ equation.

The recurrence relation (3.17) can be solved explicitly. To be able to express the solution of in a compact form it is useful to introduce a contraction operator d.

Definition 3.2 *Let* d *be a contraction operator whose action on an ordered product* $\prod\limits_{\substack{j=1\\ \longrightarrow}}^{M} v^+(\mu_j)$, $M\geq 2$, *is given by*

$$d\left(v^+(\mu_1)v^+(\mu_2)\ldots v^+(\mu_M)\right)=2\sum_{j=1}^{M-1}X^+(\mu_j)\sum_{k=j+1}^{M}\frac{(-1)^{\sigma(jk)}}{\mu_j-\mu_k}\prod_{\substack{m\neq j,k\\ \longrightarrow}}^{M}v^+(\mu_m), \tag{3.27}$$

where $\sigma(jk)$ *is the parity of the permutation*

$$\sigma : (1,2,\ldots,j,j+1,\ldots,k,\ldots,M) \to (1,2,\ldots,j,k,j+1,\ldots,M) \; .$$

The d operator can be applied on an ordered product $\prod\limits_{\overrightarrow{j=1}}^{M} v^+(\mu_j)$ consecutively several times, up to $[M/2]$, the integer part of $M/2$.

Theorem 3.1 *Explicit solution to the recurrence relation (3.17) is given by*

$$B_M(\mu_1,\ldots,\mu_M) = \prod_{\overrightarrow{j=1}}^{M} v^+(\mu_j) + \sum_{m=1}^{[M/2]} \frac{1}{m!} \, d^m \prod_{\overrightarrow{j=1}}^{M} v^+(\mu_j) \; . \tag{3.28}$$

The properties of the creation operators B_M studied in this Section will be fundamental tools in determining characteristics of the $osp(1|2)$ Gaudin model. Our primary interest is to obtain the spectrum and the eigenvectors of the generating function of integrals of motion $t(\lambda)$ (3.10).

Theorem 3.2 *The lowest spin vector* Ω_- *(3.13) is an eigenvector of the generating function of integrals of motion* $t(\lambda)$ *(3.10) with the corresponding eigenvalue* $\Lambda_0(\lambda)$

$$t(\lambda)\,\Omega_- = \Lambda_0(\lambda)\,\Omega_- \; , \quad \Lambda_0(\lambda) = \rho^2(\lambda) + \rho'(\lambda) \; . \tag{3.29}$$

Furthermore, the action of the B*-operators on the lowest spin vector* Ω_- *yields the eigenvectors*

$$\Psi(\mu_1,\ldots,\mu_M) = B_M(\mu_1,\ldots,\mu_M)\,\Omega_- \; , \tag{3.30}$$

of the $t(\lambda)$ *operator*

$$t(\lambda)\Psi(\mu_1,\ldots,\mu_M) = \Lambda(\lambda;\, \{\mu_j\}_{j=1}^M)\,\Psi(\mu_1,\ldots,\mu_M) \; , \tag{3.31}$$

with the eigenvalues

$$\Lambda(\lambda;\, \{\mu_j\}_{j=1}^M) = \Lambda_0(\lambda) + 2\rho(\lambda)\sum_{k=1}^{M} \frac{1}{\lambda-\mu_k} + 2\sum_{k<l} \frac{1}{(\lambda-\mu_k)(\lambda-\mu_l)} \; , \tag{3.32}$$

provided that the Bethe equations are imposed on the parameters $\{\mu_j\}_{j=1}^M$

$$\beta_M(\mu_j) = \rho(\mu_j) + \sum_{k\neq j}^{M} \frac{1}{\mu_j-\mu_k} = 0 \; . \tag{3.33}$$

Proof. The equation (3.29) can be checked by a direct substitution of the definitions of the operator $t(\lambda)$ and the lowest spin vector Ω_-, the equations (3.10) and (3.13), respectively.

To show the second part of the theorem, we use the equation (3.30) to express the Bethe vectors $\Psi(\mu_1,\ldots,\mu_M)$

$$t(\lambda)\Psi(\mu_1,\ldots,\mu_M) = t(\lambda)\; B_M(\mu_1,\ldots,\mu_M)\,\Omega_- \; . \tag{3.34}$$

Our next step is to use the third property of the B-operators, the equation (3.25), and the definition of the lowest spin vector Ω_- (3.13) in order to calculate the action of the operator $t(\lambda)$ on the Bethe vectors when the Bethe equations (3.33) are imposed

$$t(\lambda)B_M\Omega_- = B_M t(\lambda)\Omega_- + 2\left(\rho(\lambda)\sum_{i=1}^{M}\frac{1}{\lambda-\mu_i} + \sum_{i<j}^{M}\frac{1}{(\lambda-\mu_i)(\lambda-\mu_j)}\right)B_M\Omega_- \, . \quad (3.35)$$

We can express the first term on the right hand side since we know how the operator $t(\lambda)$ acts on the vector Ω_-, the equation (3.29),

$$t(\lambda)B_M\Omega_- = \left(\Lambda_0(\lambda) + 2\left(\rho(\lambda)\sum_{i=1}^{M}\frac{1}{\lambda-\mu_i} + \sum_{i<j}^{M}\frac{1}{(\lambda-\mu_i)(\lambda-\mu_j)}\right)\right)B_M\Omega_- \, . \quad (3.36)$$

The eigenvalue equation (3.31) as well as the expression for the eigenvalues (3.32) follow from the equation (3.36). □

Corollary 3.1 *In the Gaudin realization of the loop superalgebra given by the equations (3.6) and (3.14) the Bethe vectors* $\Psi(\mu_1,\ldots,\mu_M)$ *(3.30) are the eigenvectors of the Gaudin hamiltonians (3.15)*

$$H^{(a)}\Psi(\mu_1,\ldots,\mu_M) = E_M^{(a)}\Psi(\mu_1,\ldots,\mu_M) \, , \quad (3.37)$$

with the eigenvalues

$$E_M^{(a)} = \sum_{b\neq a}^{N}\frac{l_a\, l_b}{z_a - z_b} + \sum_{j=1}^{M}\frac{l_a}{\mu_j - z_a} \, , \quad (3.38)$$

when the Bethe equations are imposed

$$\beta_M(\mu_j) = \rho(\mu_j) + \sum_{k\neq j}^{M}\frac{1}{\mu_j-\mu_k} = \sum_{a=1}^{N}\frac{-l_a}{\mu_j - z_a} + \sum_{k\neq j}^{M}\frac{1}{\mu_j-\mu_k} = 0 \, . \quad (3.39)$$

The statement of the corollary follows from residue of the equation (3.31) at the point $\lambda = z_a$. The residue can be determined using (3.16), (3.32) and (3.29).

The eigenvalue (3.32) of the operator $t(\lambda)$ and the Bethe equations (3.33) can be obtained also as the appropriate terms in the quasi-classical limit $\eta \to 0$ of the expressions (2.8) and (2.10).

Comparing the eigenvalues $E_M^{(a)}$ (3.38) of the Gaudin hamiltonians and the Bethe equations (3.39) with the corresponding quantities of the $sl(2)$ GM [3, 4] we arrive to an interesting observation.

Remark 3.2 *The spectrum of the* $osp(1|2)$ *Gaudin model with the spins* l_a *coincides with the spectrum of the* $sl(2)$ *GM for the integer spins (cf. an analogous observation for partition functions of corresponding anisotropic vertex models in [19]).*

Remark 3.3 *The Bethe vectors are eigenstates of the global generator* h_{gl}

$$h_{gl}\Psi(\mu_1,\ldots,\mu_M) = \left(-\sum_{a=1}^{N} l_a + M\right)\Psi(\mu_1,\ldots,\mu_M) \, . \quad (3.40)$$

Moreover, these Bethe vectors are the lowest spin vectors of the global $osp(1|2)$ *since they are annihilated by the generator* v_{gl}^-

$$v_{gl}^-\Psi(\mu_1,\ldots,\mu_M)=0\,, \tag{3.41}$$

once the Bethe equations are imposed (3.39). These conclusions follow from the remark 3.2 the equations (3.23), (3.24) and the definition of the Bethe vectors (3.30).

Hence, action of the global generator v_{gl}^+ on the lowest spin vectors $\Psi(\mu_1,\ldots,\mu_M)$ generates a multiplet of eigenvectors of the operator $t(\lambda)$. One can repeat the arguments of [12, 1] to demonstrate combinatorially completeness of the constructed states.

As was pointed out already in [3] for the $sl(2)$ case, there are several modifications of the hamiltonians (3.15). One of them is the Richardson's pairing-force hamiltonian. These modifications can be formulated in the framework of universal L-operator and r-matrix formalism (3.2) [4].

Due to invariance of the r-matrix (3.1) one can add to the L-operator any element of $osp(1|2)$

$$L(\lambda)\to\tilde{L}(\lambda)=g\,Y+L(\lambda)\,, \tag{3.42}$$

preserving commutation relations (3.2). If we choose $Y=h$, then

$$\widetilde{t}(\lambda)=\tfrac{1}{2}\,\mathrm{str}\,\widetilde{L}^2(\lambda)=t(\lambda)+2g\,h(\lambda)+g^2 \tag{3.43}$$

will have the commutativity property, *i.e.* $\widetilde{t}(\lambda)\widetilde{t}(\mu)=\widetilde{t}(\mu)\widetilde{t}(\lambda)$. Hence we can take $\widetilde{t}(\lambda)$ to be the generating function of the (modified) integrals of motion

$$\widetilde{H}^{(a)} \;=\; g\,h_a+\sum_{b\neq a}\frac{c_2^{\otimes}(a,b)}{z_a-z_b}\,. \tag{3.44}$$

Notice that the global $osp(1|2)$ symmetry is now broken down to global $u(1)$. In this case the eigenstates Ψ are generated by the same B-operators, but eigenvalues and Bethe equations are slightly changed. Richardson like hamiltonian [3] can be obtained as a coefficient of $1/\lambda^2$ in the $\lambda\to\infty$ expansion

$$\widetilde{t}(\lambda) \;=\; g^2+\frac{2g}{\lambda}\,h_{gl}+\frac{1}{\lambda^2}\left(2g\sum_{a=1}^{N}z_ah_a+c_2(gl)\right)+O(\frac{1}{\lambda^3})\,. \tag{3.45}$$

Another modification can be obtained using an L-operator with bosonic and fermionic oscillator entries

$$L_{osc}(\lambda)=\begin{pmatrix}\lambda & -\gamma & 2b\\ \gamma^+ & 0 & \gamma\\ 2b^+ & \gamma^+ & -\lambda\end{pmatrix}\,, \tag{3.46}$$

where $[b\,,\,b^+]=1$, $[\gamma\,,\,\gamma^+]_+=1$ and $\gamma^2=(\gamma^+)^2=0$. It is straightforward to see that the corresponding realization of the loop superalgebra will have only two nonzero commutators. Hence, one can consider a combination of Gaudin and oscillator realizations

$$\widetilde{L}(\lambda)=L_{osc}(\lambda)+L(\lambda)\,, \tag{3.47}$$

preserving all the properties mentioned above of the corresponding B-operators with appropriate changes in hamiltonians and Bethe equations: $\beta_M(\mu_j) \to \beta_M(\mu_j) + \mu_j$.

Further modifications can be obtained considering quasi-classical limit of the quantum spin system with non-periodic boundary conditions and corresponding reflection equation.

The expression of the eigenvector of a solvable model in terms of local variables parameterized by sites of the chain or by space coordinates is known as coordinate Bethe Ansatz [3]. The coordinate representation of the Bethe vectors gives explicitly analytical dependence on the parameters $\{\mu_i\}_1^M$ and $\{z_a\}_1^N$ useful in a relation to the KZ equation (Section 4). Using the Gaudin realization (3.6) of the generators

$$v^+(\mu) = \sum_{a=1}^{N} \frac{v_a^+}{\mu - z_a} \ , \quad X^+(\mu) = \sum_{a=1}^{N} \frac{X_a^+}{\mu - z_a} \ ,$$

and the definition of the creation operators (3.28), one can get the coordinate representation of the B-operators:

$$B_M(\mu_1, \mu_2, ..., \mu_M) = \sum_{\pi} \left(v_{a_1}^+ \cdots v_{a_M}^+\right)_\pi \prod_{a=1}^{N} \varphi(\{\mu_m^{(a)}\}_1^{|\mathcal{K}_a|}; z_a) \ , \tag{3.48}$$

where the first sum is taken over ordered partitions π of the set $(1, 2, \dots, M)$ into subsets $\mathcal{K}_a$, $a = 1, 2, \dots, N$, including empty subsets with the constraints

$$\bigcup_a \mathcal{K}_a = (1, 2, \dots, M) \ , \quad \mathcal{K}_a \bigcap \mathcal{K}_b = \emptyset \quad \text{for } a \neq b \ .$$

The corresponding subset of quasimomenta

$$\left(\mu_1^{(a)} = \mu_{j_1}, \mu_2^{(a)} = \mu_{j_2}, \dots \mu_{|\mathcal{K}_a|}^{(a)} = \mu_{j_{|\mathcal{K}_a|}}; j_m \in \mathcal{K}_a\right) \ ,$$

where $\mid \mathcal{K}_a \mid$ is the cardinality of the subset $\mathcal{K}_a$, and $j_k < j_{k+1}$, entering into the coordinate wave function

$$\varphi(\{\nu_m\}_1^{|\mathcal{K}|}; z) = \sum_{\sigma \in \mathcal{S}_{|\mathcal{K}|}} (-1)^{p(\sigma)} \left((\nu_{\sigma(1)} - \nu_{\sigma(2)})(\nu_{\sigma(2)} - \nu_{\sigma(3)}) \cdots (\nu_{|\mathcal{K}|} - z)\right)^{-1} .$$

Due to the alternative sum over permutations $\sigma \in \mathcal{S}_{|\mathcal{K}|}$ this functions is antisymmetric with respect to the quasi-momenta. Finally the first factor in (3.48) $\left(v_{a_1}^+ \cdots v_{a_M}^+\right)_\pi$ means that for $j_m \in \mathcal{K}_a$ corresponding indices of $v_{a_{jm}}^+$ are equal to a so that $v_{a_{jm}}^+ = v_a^+$. One can collect these operators into product $\prod_{a=1}^{N} (v_a^+)^{|\mathcal{K}_a|}$, consequently we have an extra sign factor $(-1)^{p(\pi)}$.

This coordinate representation is similar to the representations obtained in [7, 8, 9] for the Gaudin models related to the simple Lie algebras. The Z_2-grading results in extra signs, while the complicated structure of the B_M-operators is connected with the fact that $(v_j^+)^2 = X_j^+ \neq 0$ while for $j \neq k$ v_j^+ and v_k^+ anticommute.

4 Solutions to the Knizhnik-Zamolodchikov equation

Correlation functions $\psi(z_1,\ldots,z_n)$ of a two dimensional conformal field theory satisfy the Knizhnik-Zamolodchikov equation [20]

$$\kappa\frac{\partial}{\partial z_a}\psi(z_1,\ldots,z_n) = \left(\sum_{b\neq a}\frac{Y_a^\alpha\otimes Y_b^\alpha}{z_a-z_b}\right)\psi(z_1,\ldots,z_n)\ , \tag{4.1}$$

where Y_a^α are generators of an orthonormal basis of a simple Lie algebra in a finite dimensional irreducible representation V_a and $\psi(z_1,\ldots,z_n)$ is a function of N complex variables taking values in a tensor product $\overset{N}{\underset{a=1}{\otimes}} V_a$. The operator on the right hand side of (4.1) is a Gaudin hamiltonian (1.1).

A relation between the Bethe vectors of the Gaudin model related to simple Lie algebras and the solutions to the KZ equation is well known for sometime [7, 8]. Approach used here to obtain solutions to the KZ equation corresponding to Lie superalgebra $osp(1|2)$ starting from B-vectors (3.30) is based on [7].

A solution in question is represented as a contour integral over the variables $\mu_1\ldots\mu_M$

$$\psi(z_1,\ldots z_N) \quad=\quad \oint\ldots\oint\ \phi(\vec{\mu}|\vec{z})\Psi(\vec{\mu}|\vec{z})\ d\mu_1\ldots d\mu_M\ , \tag{4.2}$$

where an integrating factor $\phi(\vec{\mu}|\vec{z})$ is a scalar function

$$\phi(\vec{\mu}|\vec{z}) \quad=\quad \prod_{i<j}^{M}(\mu_i-\mu_j)^{\frac{1}{\kappa}}\prod_{a<b}^{N}(z_a-z_b)^{\frac{l_a l_b}{\kappa}}\left(\prod_{k=1}^{M}\prod_{c=1}^{N}(\mu_k-z_c)^{\frac{-l_c}{\kappa}}\right)\ , \tag{4.3}$$

and $\Psi(\vec{\mu}|\vec{z})$ is a Bethe vector (3.30) where the Bethe equations are not imposed.

Differentiating the integrand of (4.2) one finally gets

$$\kappa\partial_{z_a}(\phi\Psi) = H^{(a)}(\phi\Psi) + \kappa\sum_{j=1}^{M}\partial_{\mu_j}\left(\frac{(-1)^j}{\mu_j-z_a}\,\phi\,\widetilde{\Psi}^{(j,a)}\right)\ . \tag{4.4}$$

A closed contour integration with respect to $\mu_1,\ldots,\mu_M$ will cancel the contribution from the terms under the sum in (4.4) and therefore $\psi(z_1,\ldots z_N)$ given by (4.2) satisfies the KZ equation. This follows from the form of the integrating factor (4.3)

$$\kappa\partial_{z_a}\phi \quad=\quad \left(\sum_{b\neq a}^{N}\frac{l_a\,l_b}{z_a-z_b}-\sum_{j=1}^{M}\frac{l_a}{z_a-\mu_j}\right)\phi = E_M^{(a)}\phi\ , \tag{4.5}$$

$$\kappa\partial_{\mu_j}\phi \quad=\quad \left(\sum_{a=1}^{N}\frac{-l_a}{\mu_j-z_a}+\sum_{j\neq k}^{M}\frac{1}{\mu_j-\mu_k}\right)\phi = \beta_M(\mu_j)\phi\ , \tag{4.6}$$

the differential identity (3.26) and commutation relations (3.25).

Conjugated Bethe vectors $(B_M\Omega_-)^*$ are entering into solution $\widetilde{\psi}(z_1,\ldots z_N)$ of the dual KZ equation

$$-\kappa\frac{\partial}{\partial z_a}\widetilde{\psi}(z_1,\ldots,z_N) \quad=\quad \widetilde{\psi}(z_1,\ldots,z_N)\ H^{(a)}\ . \tag{4.7}$$

The scalar product $\left(\widetilde{\psi}(z_1,\ldots,z_N)\ ,\ \psi(z_1,\ldots,z_N)\right)$ does not depend on $\{z_j\}_1^N$ and its quasi-classical limit $\kappa\to 0$ gives the norm of the Bethe vectors [9] due to the fact that the stationary points of the contour integrals for $\kappa\to 0$ are solutions to the Bethe equations $\partial S/\partial\mu_j=\beta_M(\mu_j)\phi$,

$$S(\vec{\mu}|\vec{z})=\kappa\ln\phi=\sum_{a<b}^{N}l_a l_b\ln(z_a-z_b)+\sum_{i<j}^{M}\ln(\mu_i-\mu_j)-\sum_{a=1}^{N}\sum_{j=1}^{M}l_a\ln(z_a-\mu_j)\ . \tag{4.8}$$

According to the observation in the end of Section 3 analytical properties of the Bethe vectors of the $osp(1|2)$ Gaudin model coincide with the analytical properties of the $sl(2)$ Gaudin model. Thus, the expression for the norm of the Bethe vectors (3.30) obtained as the first term in the asymptotic expansion $\kappa\to 0$ coincides also

$$\parallel\Psi(\mu_1,\ldots\mu_M;\{z_a\}_1^N)\parallel^2=\det\left(\frac{\partial^2 S}{\partial\mu_j\,\partial\mu_k}\right)\ , \tag{4.9}$$

$$\frac{\partial^2 S}{\partial\mu_j^2}\ =\ \sum_{a=1}^{N}\frac{l_a}{(\mu_j-z_a)^2}-\sum_{k\neq j}^{M}\frac{1}{(\mu_j-\mu_k)^2}\ ,\quad \frac{\partial^2 S}{\partial\mu_j\,\partial\mu_k}=\frac{1}{(\mu_j-\mu_k)^2}\ ,\quad \text{for } j\neq k\ . \tag{4.10}$$

Finally we notice that the modification of the Gaudin hamiltonians we mentioned at the end of the previous Section can be transfered to the corresponding modification of the KZ equations. Both modifications are related with extra factors in the integrating scalar function (4.3)

5 Conclusion

The Gaudin model corresponding to the simplest non-trivial Lie superalgebra $osp(1|2)$ was studied. A striking similarity between some of the most fundamental characteristics of this system and the $sl(2)$ GM was found. Although explicitly constructed creation operators B_M (3.28) of the Bethe vectors are complicated polynomials of the generators $v^+(\lambda)$ and $X^+(\lambda)$, the coordinate form of the eigenfunctions differs only in signs from the corresponding states in the case of $sl(2)$ model. Moreover, the eigenvalues and the Bethe equations coincide, provided that the $sl(2)$ Gaudin model with integer spins is considered.

Let us point out that using the method proposed in this paper one can construct explicitly creation operators of the Gaudin models related to trigonometric Izergin-Korepin r-matrix [16] and trigonometric $osp(1|2)$ r-matrix [21]. Similarly to the simple Lie algebra case solutions to the KZ equation were constructed from the Bethe vectors using algebraic properties of the creation operators B_M and the Gaudin realization of the loop superalgebra $\mathcal{L}_+(osp(1|2))$. This interplay enabled us to determine the norm of eigenfunctions (4.9).

The difficult problem of correlation function calculation for general Bethe vectors

$$\mathcal{C}\left(\{\nu_j\}_1^M;\{\mu_i\}_1^M;\{\lambda_k\}_1^K\right)=\left(\Omega_-,B_M^*(\nu_1,\ldots\nu_M)\prod_{k=1}^{K}h(\lambda_k)B_M(\mu_1,\ldots\mu_M)\Omega_-\right)$$

was solved nicely for the $sl(2)$ Gaudin model in [5] using the Gauss factorization of the loop algebra group element and the Riemann-Hilbert problem. The study of this problem

for the $osp(1|2)$ GM is in progress and the following expression for the scalar product of the Bethe states is conjectured (cf. [5])

$$(\Omega_- \ , \ B_M^*(\nu_1, \dots \nu_M) B_M(\mu_1, \dots \mu_M)\Omega_-) = \sum_{\sigma \in \mathcal{S}_M} (-1)^{p(\sigma)} \det \mathcal{M}^\sigma \ ,$$

where the sum is over symmetric group $\mathcal{S}_M$ and $M \times M$ matrix $\mathcal{M}^\sigma$ is given by

$$\begin{aligned}
\mathcal{M}^\sigma_{jj} &= \frac{\rho(\mu_j) - \rho(\nu_{\sigma(j)})}{\mu_j - \nu_{\sigma(j)}} - \sum_{k \neq j}^{M} \frac{1}{(\mu_j - \mu_k)(\nu_{\sigma(j)} - \nu_{\sigma(k)})} \ , \\
\mathcal{M}^\sigma_{jk} &= \frac{1}{(\mu_j - \mu_k)(\nu_{\sigma(j)} - \nu_{\sigma(k)})} \ , \quad \text{for } j, k = 1, 2, \dots M .
\end{aligned}$$

6 Acknowledgements

We acknowledge useful discussions and communications with N. Yu. Reshetikhin, V. O. Tarasov and T. Takebe. This work was supported by the grant PRAXIS XXI/BCC/22204/99, INTAS grant N 99-01459 and FCT project SAPIENS-33858/99.

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